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Sufficient Conditions for Univalent Functions

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Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in N = \{1,2,3,\cdots\})$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

Ozaki, Ono and Umezawa [5] proved the following result.

Theorem A. If $f(z) \in A(1)$ and $|f''(z)| < 1$ in U, then $f(z)$ is univalent in U.

Recently, Nunokawa, Kwon and Cho [2] have proved the following result.

Theorem B. Let $p \geqq 2$. If $f(z) \in A(p)$ and suppose that

$$|f^{(p+1)}(z)| < 2(p!) \quad \text{in } U,$$

then $f(z)$ is p-valent in U.

Applying Schwarz's lemma, it is easily confirmed that Theorem A and the following Theorem A' are equivalent.

Theorem A'. If $f(z) \in A(1)$ and $|zf''(z)| < 1$ in U, then $f(z)$ is univalent in U.

We here obtain the following result concerned with Theorem A and Theorem A'.

Theorem 1. Let $f(z) \in A(1)$ and suppose that

(1) $$\left|\arg\left(zf''(z) + \frac{1}{2}\right)\right| < \pi \quad \text{in } U.$$

Then $f(z)$ is univalent in U.

Proof. Let us put $p(z) = f'(z)$ and

$$\phi(z) = \frac{1 - p(z)}{1 + p(z)}.$$

Then, if there exists a point $z_0 \in U$ such that

$$\mathrm{Re}\ p(z) > 0 \quad \text{for } |z| < |z_0|$$

$$\mathrm{Re}\ p(z_0) = 0,$$

then we have that $\phi(0) = 0$, $|\phi(z)| < 1$ for $|z| < |z_0|$ and $|\phi(z_0)| = 1$.

Applying the same method as the proof of [3, Theorem 1], we have that $z_0 p'(z_0)$ is a real number and

$$z_0 p'(z_0) \leqq -\frac{1}{2}\left(1 + |p(z_0)|^2 \right) \leqq -\frac{1}{2}.$$

This contradicts (1). Therefore we have

$$\operatorname{Re} f'(z) > 0 \quad \text{in } U.$$

From Noshiro's theorem [1], f(z) is univalent in U. This completes our proof.

Concerning a sufficient conditions for the univalency of f(z), it is trivial that Theorem 1 is better than Theorem A and Theorem A'.

Theorem 2. Let $f(z) \in A(p)$ and suppose that

$$\left| \arg \left(zf^{(p+1)}(z) + \frac{p!}{2} \right) \right| < \pi \quad \text{in } U.$$

Then f(z) is p-valent in U.

Proof. Applying the same method as the proof of Theorem 1, we have

$$\operatorname{Re} f^{(p)}(z) > 0 \quad \text{in } U.$$

Then, from Ozaki's theorem [4, Theorem 2], f(z) is p-valent in U. This completes our proof.

Theorem 2 is also better than Theorem B.

References

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